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ON THE SHORTEST ROUTE THROUGH A NETWORK

George B. Dantzig

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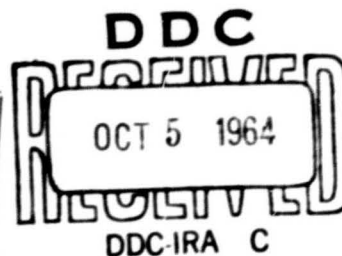
April 12, 1958 ✓

Revised April 29, 1959

Approved for OTS release

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SUMMARY

The chief feature of the method is that it fans out from the origin working out the shortest path to one new node from the origin and never having to backtrack. No more than $n(n-1)/2$ comparisons are needed to find the shortest route from a given origin to all other nodes and possibly less between two fixed nodes.

Except for details and bias of various authors towards a particular brand of proof, this problem has been solved the same way by many authors. This paper refines these proposals to give what is believed to be the shortest procedure for finding the shortest route when it is little effort to arrange distances in increasing order by nodes or to skip consideration of arcs into nodes whose shortest route to the origin has been determined earlier in the computation.

In practice the number of comparisons is much less than indicated bounds because all arcs leading to nodes previously evaluated are deleted from further consideration. A further efficiency can be achieved in the event of ties by including least distances from origin to many nodes simultaneously during the fanning out process. However, these are shown as separate steps to illustrate the underlying principle.

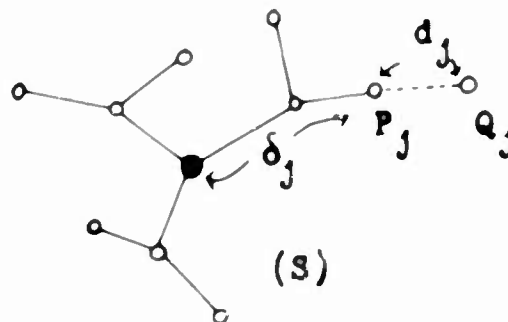
ON THE SHORTEST ROUTE THROUGH A NETWORK

George B. Dantzig

The purpose of this paper is to give what is believed to be the shortest procedure for obtaining the shortest route from a given origin to all other nodes in the network or to a particular destination point. The method can be interpreted as a slight refinement of those reported by Bellman, Moore, Ford, and the author in [1], [2], [3], [4], and those proposed by Gale and Fulkerson in informal conversations. It is similar to Moore's method of fanning out from the origin. However, its special feature — which is believed to be new is that the fanning out is done one point at a time and the distance assigned is final.

It is assumed (a) that one can write down without effort for each node the arcs leading to other nodes in increasing order of length and (b) that it is no effort to ignore an arc of the list if it leads to a node that has been reached earlier. It will be shown that no more than $n(n - 1)/2$ comparisons are needed in an n node network to determine the shortest routes from a given origin to all other nodes and less than half this number for a shortest route to particular node. The basic idea is as follows:

Suppose at some stage k in the computing process the shortest distances to k of the nodes from some origin are known as well as the paths. Call the set of these points S .



- (1) Let P_j be one of the nodes in S ,
- (2) let δ_j be its least distance to the origin \bullet ,
- (3) let Q_j be the nearest node to P_j not in S ,
- (4) let d_j be the distance from P_j to Q_j . Choose as the $k + 1$ point, Q_s , where s satisfies

$$\delta_s + d_s = \text{Min}(\delta_j + d_j) \quad j = 1, 2, \dots, k$$

The minimum distance of Q_s to the origin is $\delta_s + d_s$ and the best path to the origin is via P_s . The reason is obvious for if the best path from Q_s were via some other j in S or via several other points not in S and then via some other j in S , then the distance is at least $\delta_j + d_j \geq \delta_s + d_s$. In case of ties for minimum, several such nodes Q_s could be determined at the same time and the process made more efficient.

It will be noted that the minimum requires only k comparisons for a decision as to the $k + 1$ st point. Hence in an n node network no more than

$$1 + 2 + \dots + (n - 1) = n(n - 1)/2$$

comparisons are needed.

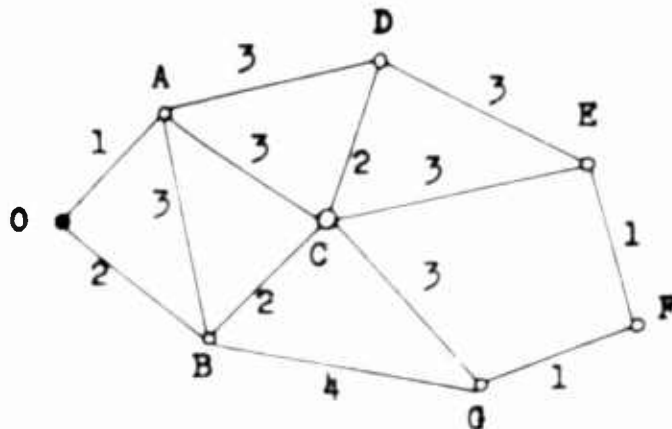
In practice the number of comparisons can be considerably less than this bound because after several stages one or more of the nodes in S have only arcs leading to points in S [in the 8 node example below only a total of 16 comparisons was needed instead of $7 \times 8/2 = 28$ comparisons].

If the problem is to determine only the shortest path from a given origin to a given terminal, the number of comparisons may often be reduced by fanning out both from the origin and the terminal simultaneously, — adding alternatively one point at a time to sets S about the origin and S' about the terminal.

Once the shortest distance from a node to origin is evaluated the node is conceptually connected directly to the origin by a hypothetical arc with the specified shortest distance and disconnected from all arcs leading to other nodes evaluated earlier. Nodes whose shortest distance to the terminal which have been determined are similarly treated. Once the origin is reached by either fanning system the process terminates.

EXAMPLE

Distances on links of the network are as indicated.



Arrange in ascending order the nodes by distances to a given

node

(0)	(A)	(B)	(C)	(D)	(E)	(F)	(G)
OA-1	AB-3	BC-2	CB-2	DC-2	EF-1	FE-1	GF-1
OB-2	AC-3	BA-3	CD-2	DA-3	EC-3	FG-1	GC-3
	AD-3	BG-4	CA-3	DE-3	ED-3		GB-4
			CG-3				
			CE-3				

Step 1. Choose path OA; place its distance 1 above A column, delete all arcs into A.

Step 2. Compare OB-2 and AB-(3 + 1) and choose path OB; place its distance 2 above B column, delete all arcs into B

Step 3. Compare AC-(3 + 1), AD-(3 + 1), BC-(2 + 2), and because of ties choose paths AC (or BC) and AD; place distance 4 above C column, delete all arcs into C and D

Step 4. Compare BG-(4 + 2), CG-(3 + 4), DE-(3 + 4) and choose

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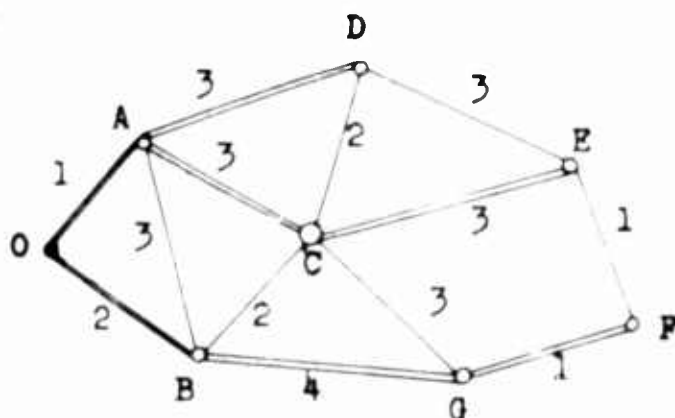
path BG, place its distance 6 above G column, delete all arcs into G

Step 5. Compare $CE-(3+4)$, $DE-(3+4)$, $GP-(1+6)$ and choose path CE (or DE), GP; place distance 7 above E and P columns, delete all arcs into E and P.

Step 6. Compare $EP-(1+7)$, $GP-(1+6)$ and choose path GP, place its distance 7 above P column, delete all arcs into P.

Because of ties many of the steps were carried on simultaneously.

The shortest paths from the origin to other nodes are along paths OA, OB, AC, AD, BG, CE, GP with alternative BC for AC and DE for CE.



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